LECTURE NOTES

PROGRAMME – BCA

SEMESTER- IV

DISCRETE MATHEMATICS (BCA-401)

UNIT I

Set Theory: Introduction, Combination of sets, Multisets, Ordered pairs. Proofs of some general identities on sets.

Relation: Relations on sets, Types of relations in a set, Properties of relations, Composition of relations, Representation of relations, Closures of relations.

Function: Types of functions, Composition of functions, Recursively defined function.

SET THEORY AND RELATION

Unit Structure

3.0 OBJECTIVES:

1. Definition and examples of sets.
2. Basic operations and diagrammatic representation of sets.
3. Definition of relations and diagraphs
4. Concept of partition and its relationship with equivalence relation.

3.1 INTRODUCTION:

In the school, we have already studied sets along with the properties of the sets. In this chapter, we revise the concept and further, discuss the concept of an algebraic property called relation.

Set Theory, branch of mathematics concerned with the abstract properties of sets, or collections of objects. A set can be a physical grouping, such as the set of all people present in a room; or a conceptual aggregate, such as the set of all British prime ministers, past and present. Each of these sets is defined by a property that its members share, but it is possible for a set to be a completely arbitrary collection.

Set theory was first given formal treatment by the German mathematician Georg Cantor in the 19th century. The set concept is one of the most basic in mathematics, explicitly or implicitly, in every area of pure and applied mathematics, as well as Computer science.

Relationships between elements of sets occur in many contexts. We deal with many relationships such as student's name and roll no., teacher and her specialisation, a person and a relative (brother – sister, mother – child etc.) In this section, we will discuss mathematical approach to the relation. These have wide applications in Computer science (e.g. relational algebra)

3.2. DEFINITIONS AND REPRESENTATION OF SETS:

Definition 3.2.1: Set is an unordered collection of objects.
The object in a set is called as an element or member.

We denote sets by capital letters such as A, B, C and elements by small letters. Typically sets are described by two methods

i. Roster or list method:
   In this method, all the elements are listed in braces. E.g.
   A = {2, 3, 5, 7, 11, 13 }
   N = { 2, 4, 6, ... }

ii. Set-Builder method:
   In this method, elements are described by the property they satisfy. E.g.
   $A = \{ x : x$ is a prime number less than 15}
   $B = \{ x : x = 2n, n \in \mathrm{N} \}$

Definition 3.2.2: A set containing no element is called as an **empty set**. E.g. Set of even prime numbers greater than 10.
Empty set is denoted by { } or ϕ.

Definition 3.2.3: A set *A* is said to be a **subset** of set *B*, if every element of *A* is also an element of *B*. It is denoted by '$\subseteq$ '
$A \subseteq B$. E.g. A = {1, 2, 3, 4 } and B = { 1, 2, 3, 4, 7, 8 } Then A $\subseteq$ B.

Definition 3.2.4: A set *A* is said to be a **superset** of set *B*, if *B* is a subset of *A*. It is denoted by $A \supseteq B$.

Definition 3.2.5: A set is *A* is said to be a **proper subset** of *B*, if *A* is a subset of *B* and there is at least one element in *B*, which is not an element of *A*. Set A explained in Definition 3.2.3, is a proper subset of *B*.

Definition 3.2.6: A set which contains all objects under consideration is called as **Universal** set and is denoted by *U*.

Note: Two sets are said to be equal if and only if they have same elements. E.g. If $A = \{2, 5, 7, 9\}$ and $B = \{5, 2, 7, 9\}$, then A and B are equal.

Now we shall discuss various operations on sets. For this discussion, let U be universal set and let A and B be two subsets of U.

Definition 3.2.7: Set of all elements in A or in B or in both, is defined as **union** of A and B and is denoted by $A \cup B$.

E.g. If $A = \{1, 2, 3, 5, 7\}$ and $B = \{2, 5, 10\ 11\}$, then
$A \cup B = \{1, 2, 3, 5, 7, 10, 11\}$

Definition3.2.8: Set of all elements, that are common in A as well as in B, is defined as **intersection** of A and B and is denoted by $A \cap B$.

E.g. If $A = \{1, 2, 3, 5, 7\}$ and $B = \{2, 5, 10, 11\}$, then
$A \cap B = \{2, 5\}$.

Definition 3.2.9: Set of all elements, that are in A, but not in B, is called as difference between A and B and denoted by
$A - B$. E.g. If $A = \{1, 4,7,8,9\}$ and $B = \{4,9,11,13\}$ then, $A - B = \{1,7, 8\}$.

Definition 3.2.10: The total number of elements in a set is called as **cardinality** of a set. E.g. If $A = \{2, 3, 5, 7, 11, 13\}$ then, Cardinality of A, denoted by $|A|$, is 6. If a set is infinite, then its cardinality is infinity.

Definition 3.2.11: If U is a universal set and A is its subset, then complement of A, denoted by A^C, is all elements of U, that are not in A. E.g. If $U = \{ x : x \in \mathrm{N}, x \le 15 \}$ and
$A = \{\mathrm{x} : x \in \mathrm{U} \text{ and } 3 \mid x \}$, then $A^C = \{ 1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}$.

Definition 3.2.12: A power set of a set A, denoted by $P(A)$, is set of all subsets of A. E.g. If $A = \{ 1, 2, 3 \}$, then,
$P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\},\{1, 2, 3\}\}$.

Note: If number of elements in A is n, then the number of elements in the power set of A is 2^n.

Definition 3.2.13: Let A and B be two sets. The product set of A and B (or Cartesian product of A and B), denoted by
$A \times B$, is set of all ordered pairs from A and B. Thus,
$A \times B = \{(a, b): a \in A, b \in B\}$.
E.g. Let $A = \{ 1, 2, 3 \}$ and $B = \{ 4, 5\}$ then
$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

3.3 DIAGRAMMATIC REPRESENTATION OF A SET:

British mathematician, John Venn, devised a simple way to represent set theoretic operations diagrammatically. These diagrams are named after him as Venn Diagrams.

Universal set is represented by a rectangle and its subsets using a circle within it.

In the following figures, basic set theoretic operations are represented using Venn diagrams.

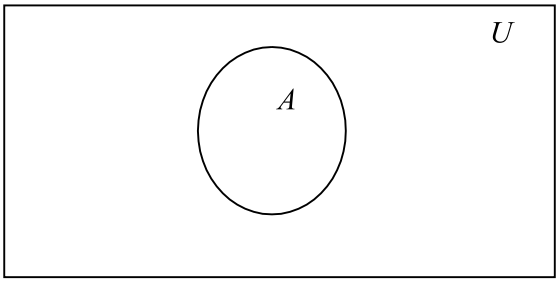

Figure 3.1: *A* is a subset of universal set *U.*

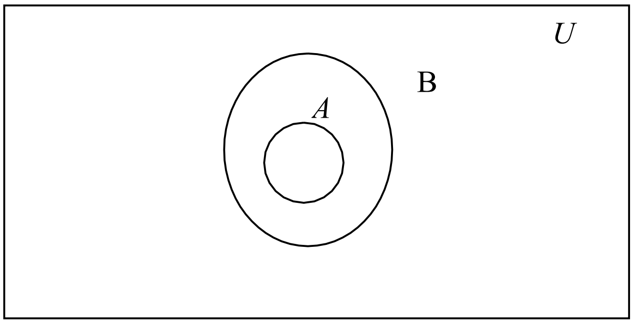

Figure 3.2: $A \subseteq B$

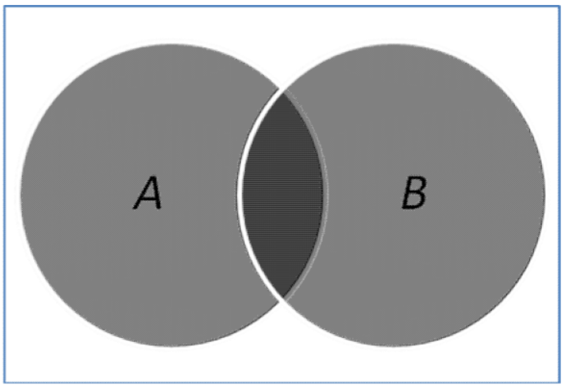

Figure 3.3: $A \cup B$ **: Entire shaded region**
$A \cap B$ **: Dark gray shaded region**

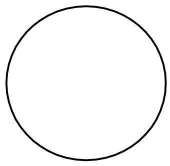

Figure 3.4: A^C, the shaded region

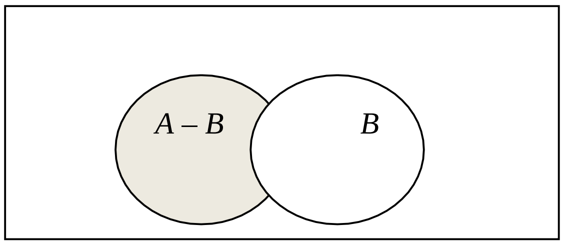

Figure 3.5: $A - B$, the shaded region

3.4 THE ALGEBRA OF SETS:

The following statements are basic consequences of the above definitions, with A, B, C, ... representing subsets of a universal set U.

1. $A \cup B = B \cup A$. (Union is commutative)
2. $A \cap B = B \cap A$. (Intersection is commutative)
3. $(A \cup B) \cup C = A \cup (B \cup C)$. (Union is associative)
4. $(A \cap B) \cap C = A \cap (B \cap C)$. (Intersection is associative)
5. $A \cup \phi = A$.
6. $A \cap \phi = \phi$.
7. $A \cup U = U$.
8. $A \cap U = A$.
9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.(Union distributes over intersection)
10. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (Intersection distributes over union)
11. $A \cup A^C = U$.
12. $A \cap A^C = \phi$.
13. $(A \cup B)^C = A^C \cap B^C$. (de' Morgan's law)
14. $(A \cap B)^C = A^C \cup B^C$. (de' Morgan's law)
15. $A \cup A = A \cap A = A$.
16. $(A^C)^C = A$.
17. $A - B = A \cap B^C$.
18. $(A - B) - C = A - (B \cup C)$.
19. If $A \cap B = \phi$, then $(A \cup B) - B = A$.
20. $A - (B \cup C) = (A - B) \cap (A - C)$.

This algebra of sets is an example of a Boolean algebra, named after the 19th-century British mathematician George Boole, who applied the algebra to logic. The subject later found applications in electronics.

3.5 THE COMPUTER REPRESENTATION OF SETS:

There are various ways to represent sets using a computer. Modern programming languages, such as JAVA, have predefined collection class to represent the set. In such class, we need to insert the set elements and there are various class operations defined for the algebraic operations on the set.

In this section, we shall present a method for storing elements using the arbitrary ordering of the elements of a universal set.

Assume that the universal set U is finite (and of reasonable size so that the number of elements in U are not larger than the memory size). First, specify the arbitrary ordering of elements of U, such as $a_1, a_2, ..., ..., a_n$. Represent a subset A of U with the bit string of length n, where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 otherwise.

E.g. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and A be subset of U containing elements that are multiples of 3 or 5. Thus,

$A = \{3, 5, 6, 9, 10\}$. We shall represent elements of U as per the order given in the above set. Then, A represents a bit string 0010110011.

With this, we have completed basic discussion on set theory and now is the time to check the understanding for the same.

3.6 RELATIONS:

Relationship between elements of sets is represented using a mathematical structure called relation. The most intuitive way to describe the relationship is to represent in the form of ordered pair. In this section, we study the basic terminology and diagrammatic representation of relation.

Definition 3.6.1: Let A and B be two sets. A **binary relation** from A to B is a subset of $A \times B$.

Note 3.6.1: If A, B and C are three sets, then a subset of $A\times B\times C$ is known as ternary relation. Continuing this way a subset of $A_1\times A_2\times ...\times A_n$ is known as n – ary relation.

Note3.6.2: Unless or otherwise specified in this chapter a relation is a binary relation.

Let A and B be two sets. Suppose R is a relation from A to B (i.e. R is a subset of $A \times B$). Then, R is a set of ordered pairs where each first element comes from A and each second element from B. Thus, we denote it with an ordered pair (a, b), where $a \in A$ and $b \in B$. We also denote the relationship with $a\ R\ b$, which is read as a related to b. The ***domain*** of R is the set of all first elements in the ordered pair and the ***range*** of R is the set of all second elements in the ordered pair.

Example 3.1: Let $A = \{ 1, 2, 3, 4 \}$ and $B = \{ x, y, z \}$. Let $R = \{(1, x), (2, x), (3, y), (3, z)\}$. Then R is a relation from A to B.

Example 3.2: Suppose we say that two countries are adjacent if they have some part of their boundaries common. Then, "is adjacent to", is a relation R on the countries on the earth. Thus, we have, (India, Nepal) $\in R$, but (Japan, Sri Lanka) $\notin R$.

Example 3.3: A familiar relation on the set **Z** of integers is "m divides n". Thus, we have, $(6, 30) \in R$, but $(5, 18) \notin R$.

Example 3.4: Let A be any set. Then $A \times A$ and ϕ are subsets of $A \times A$ and hence they are relations from A to A. These are known as universal relation and empty relation, respectively.

Note 3.6.3: As relation is a set, it follows all the algebraic operations on relations that we have discussed earlier.

Definition 3.6.2: Let R be any relation from a set A to set B. The **inverse** of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs, when reversed, belong to R. That is:
$R^{-1} = \{(b, a) : (a, b) \in R\}$

Example 3.5: Inverse relation of the relation in example 1.1 is, $R^{-1} = \{(x, 1), (x, 2), (y, 3), (z, 3)\}$.

3.7 REPRESENTATION OF RELATIONS:

Matrices and graphs are two very good tools to represent various algebraic structures. Matrices can be easily used to represent relation in any programming language in computer. Here we discuss the representation of relation on finite sets using these tools.

Consider the relation in Example 3.1.

	x	y	z
1	1	0	0
2	1	0	0
3	0	1	1
4	0	0	0

Fig. 3.1

Thus, if *a R b*, then we enter 1 in the cell (*a*, *b*) and 0 otherwise.
Same relation can be represented pictorially as well, as follows:

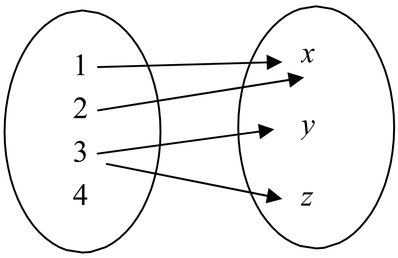

Fig 3.2

Thus, two ovals represent sets *A* and *B* respectively and we draw an arrow from $a \in A$ to $b \in B$, if *a R b*.

If the relation is from a finite set to itself, there is another way of pictorial representation, known as **diagraph**.

For example, let A = {1, 2, 3, 4} and R be a relation from A to itself, defined as follows:
R = {(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)}
Then, the diagraph of R is drawn as follows:

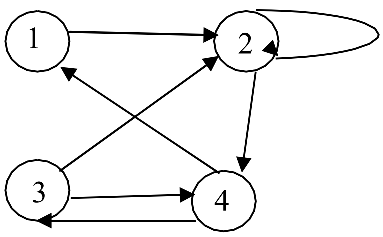

Fig 3.3

The directed graphs are very important data structures that have applications in Computer Science (in the area of networking).

Definition 3.7.1: Let A, B and C be three sets. Let R be a relation from A to B and S be a relation from B to C. Then, composite relation R°S, is a relation from A to C, defined by,
a(R°S)*c*, if there is some b $\in$ B, such that *a R b* and *b bsc*.

Example 3.6: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$,
$C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$.

Pictorial representation of the relation in Example 3.6 can be shown as below (Fig 1.4).

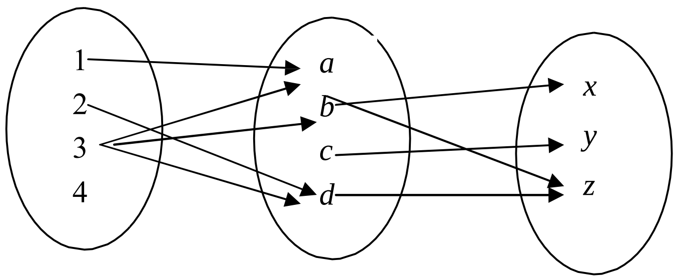

Fig 3.4

Thus, from the definition of composite relation and also from Fig 3.4, $R \circ S$ will be given as below.

$R \circ S = \{(2, z), (3, x), (3, z)\}$.

There is another way of finding composite relation, which is using matrices.

Example 3.7: Consider relations R and S in Example 3.6. Their matrix representations are as follows.

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consider the product of matrices M_R and M_S as follows:

$$M_R M_S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Observe that the non-zero entries in the product tell us which elements are related in $R \circ S$. Hence, $M_R M_S$ and $M_{R \circ S}$ have same non-zero entries.

3.8 TYPES OF RELATIONS:

In this section, we discuss a number of important types of relations defined from a set A to itself.

Definition 3.8.1: Let R be a relation from a set A to itself. R is said to be ***reflexive***, if for every $a \in A$, $a\ R\ a$ (a is related to itself).

Example 3.8: Let A = {a, b, c, d} and R be defined as follows:
$R = \{(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)\}$.
R is a reflexive relation.

Example 3.9: Let A be a set of positive integers and R be a relation on it defined as, $a\ R\ b$ if "a divides b". Then, R is a reflexive relation, as a divides to itself for every positive integer a.

Note 3.8.1: If we draw a diagraph of a reflexive relation,
then all the vertices will have a loop. Also if we represent reflexive relation using a matrix, then all its diagonal entries will be 1.

Definition 3.8.2: Let R be a relation from a set A to itself. R is said to be ***irreflexive***, if for every $a \in A$, $a\ R\ a$ (a is not related to itself).

Example 3.10: Let A be a set of positive integers and R be a relation on it defined as, $a\ R\ b$ if "a is less than b". Then, R is an irreflexive relation, as a is not less than itself for any positive integer a.

Example 3.11: Let A = {a, b, c, d} and R be defined as follows:
$R = \{(a, a), (a, c), (b, a), (b, d), (c, c), (d, c), (d, d)\}$.
Here R is neither reflexive nor irreflexive relation as b is not related to itself and a, c, d are related to themselves.

Note 3.8.2: If we draw a diagraph of an irreflexive relation,
then no vertex will have a loop. Also if we represent irreflexive relation using a matrix, then all its diagonal entries will be 0.

Definition 3.8.3: Let R be a relation from a set A to itself. R is said to be ***symmetric***, if for $a, b \in A$, if $a\ R\ b$ then $b\ R\ a$.

Definition 3.8.4: Let R be a relation from a set A to itself. R is said to be ***anti-symmetric***, if for $a, b \in A$, if $a\ R\ b$ and $b\ R\ a$, then $a = b$. Thus, R is not anti-symmetric if there exists $a, b \in A$ such that $a\ R\ b$ and $b\ R\ a$ but $a \neq b$.

Example 3.13: Let A = {1, 2, 3, 4} and R be defined as:
$R = \{(1, 2), (2, 3), (2, 1), (3, 2), (3, 3)\}$, then R is symmetric relation.

Example 3.14: An equality (or "is equal to") is a symmetric relation on the set of integers.

Example 3.15: Let *A* = {*a*, *b*, *c*, *d*} and *R* be defined as:
R = {(*a*, *b*), (*b*, *a*), (*a*, *c*), (*c*, *d*), (*d*, *b*)}. *R* is not symmetric, as *a R c* but $c\ \underline{R}\ a$. *R* is not anti-symmetric, because *a R b* and
b R a, but $a \neq b$.

Example 3.16: The relation "less than or equal to (≤)", is an anti-symmetric relation.

Definition 3.8.5: Let R be a relation defined from a set A to itself. For a, b $\in$ A, if a R b, then $b\ \underline{R}\ a$, then R is said to be ***asymmetric*** relation.

Example 3.17: Let A = {a, b, c, d} and R be defined as:
R = {(a, b), (b, c), (b, d), (c, d), (d, a)}. R here is asymmetric relation. Because $a\ \underline{R}\ b$ but $b\ \underline{R}\ a$, $b\ \underline{R}\ c$ but $c\ \underline{R}\ b$ and so on.

Example 3.18: Relation "is less than (<)", defined on the set of all real numbers, is an asymmetric relation.

Definition 3.8.6: Let *R* be a relation defined from a set *A* to itself. *R* is said to ***transitive***, if for *a*, *b*, $c \in A$, *a R b* and *b R c*, then *a R c*.

Example 3.19: Let *A* = {*a*, *b*, *c*, *d*} and *R* be defined as follows: *R* = {(*a*, *b*), (*a*, *c*), (*b*, *d*), (*a*, *d*), (*b*, *c*), (*d*, *c*)}. Here *R* is transitive relation on *A*.

Example 3.20: Relation "*a* divides *b*", on the set of integers, is a transitive relation.

Definition 3.8.7: Let *R* be a relation defined from a set *A* to itself. If *R* is reflexive, symmetric and transitive, then *R* is called as **equivalence** relation.

Example 3.21: Consider the set *L* of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other. This relation is an equivalence relation.

Example 3.22: Let *m* be a fixed positive integer. Two integers, *a*, *b* are said to be congruent modulo *m*, written as: $a \equiv b \pmod{m}$, if *m* divides *a* – *b*. The congruence relation is an equivalence relation.

Example 3.23 : Let $A=\{2,3,4,5\}$ and let $R=\{(2,3),(3,3),(4,5),(5,1)\}$.
Is R symmetric, asymmetric or antisymmetric?
Solution :

a) R is not symmetric, since $(2,3)\in R$, but $(3,2)\notin R$,

b) R is not asymmetric since $(3,3)\in R$

c) R is antisymmetric since if $a \neq b$ either

$$
\begin{array}{ccc}
(a,b)\notin R & or & (b,a)\notin R \\
2\neq 3 & , & (3,2)\notin R \\
3\neq 4 & & (3,4)\notin R \\
4\neq 5 & & (5,4)\notin R \\
5\neq 2 & & (2,5)\notin R
\end{array}
$$

Example 3.24 : Determine whether the relation R on a set A is reflenive, irreflenire, symmetric, asymmetric antisymmetric or transitive.

I) A = set of all positive integers, a R b iff $|a-b|\leq 2$.

[Dec - 02, Nov.-06, May - 07]

Solution :

1) R is reflexive because $|a-a|=0<2, \forall\, a\in A$

2) R is not irreflexive because $|1-1|=0<2$ for $1\in A$ ($\therefore$A is the set of all positive integers.)

3) R is symmetric because $|a-b|\leq 2 \Rightarrow |b-a|\leq 2 \;\therefore a\,R\,b \Rightarrow b\,R\,a$

4) R is not asymmetric because $|5-4|\leq 2$ and we have $|4-5|\leq 2$
$\therefore 5\,R\,4 \Rightarrow 4\,R\,5$

5) R is not antisymmetric because $1\,R\,2$ & $2\,R\,1$ $1\,R\,2 \Rightarrow |1-2|\leq 2$ & $2\,R\,1 \Rightarrow |2-1|\leq 2$. But $1\neq 2$

6) R is not transitive because 5 R 4, 4 R 2 but 5 $\underline{R}$ 2

II) $A=Z^{+}, a\,R\,b$ iff $|a-b|=2$ [May - 05]

Solution :

As per above example we can prove that R is not reflexive, R is irrflexive, symmetric, not asymmetric, not antisymmetric & not transitive

III) Let A = {1, 2, 3, 4} and R {(1,1), (2,2), (3,3)} [Dec. - 04]

1) R is not reflexive because $(4,4)\notin R$

2) R is not irreflexive because $(1,1)\notin R$

3) R is symmetric because whenever a R b then b R a.

4) R is not asymmetric because $|R| \Rightarrow |R|$

5) R is antisymmetric because $2\,R\,2, 2\,R\,2 \Rightarrow 2=2$

6) R is transitive.

IV) Let $A=Z^{+}, a\,R\,b$ iff GCD (a, b) = 1 we can say that a and b are relatively prime. [Apr. 04, Nov. 05]

1) R is not reflexive because $(3,3)\neq 1$ it is 3. $\therefore (3,3)\notin R$

2) R is not irreflexive because (1, 1) = 1

3) R is symmetric because for $(a,b)=1 \Rightarrow (b,a)=1$. $\therefore a\ R\ b \rightarrow b\ R\ a$

4) R is not asymmetric because (a, b) = 1 then (b, a) = 1. $\therefore a\ R\ b \rightarrow b\ R\ a$

5) R is not antisymmetric because 2 R 3 and 3 R 2 but $2 \neq 3$.

6) R is not transitive because 4 R 3, 3 R 2 but 4 $\underline{R}$ 2 because (4,2) = G.C.D. (4,2) = $2 \neq 1$.

V) A = Z a R b iff $a \leq b+1$ [May 03, May 06]

1) R is reflexive because $a \leq a+1 \ \forall\ a \in| A$.

2) R is not irreflexive because $0 \leq 0+1$ for $O \in A$.

3) R is not symmetric because for $2 \leq 5+1$ does not imply $5 \leq 2+1$.

4) R is not asymmetric because for (2,3) $\in$ R and also (3,2) $\in$R.

5) R is not antisymmetric because 5 R 4 and 4 R 5 but $4 \neq 5$.

6) R is not transitive because (6,45) $\in$ R, (5,4) $\in$ R but (6,47) $\notin$ R.

3.9 RELATIONS AND PARTITION:

In this section, we shall know what partitions are and its relationship with equivalence relations.

Definition 3.8.1: A partition or a quotient set of a non-empty set *A* is a collection P of non-empty sets of *A*, such that

(i) Each element of *A* belongs to one of the sets in P.

(ii) If A_1 and A_2 are distinct elements of P, then
$A_1 \cap A_2 = \phi$.

The sets in P are called the blocks or cells of the partition.

Example 3.23: Let *A* = {1, 2, 3, 4, 5}. The following sets form a partition of *A*, as $A = A_1 \cup A_2 \cup A_3$ and
$A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.
$A_1 = \{1, 2\}$; $A_2 = \{3, 5\}$; $A_3 = \{4\}$.

Example 3.24: Let *A* = {1, 2, 3, 4, 5, 6}. The following sets do not form a partition of *A*, as $A = A_1 \cup A_2 \cup A_3$ but
$A_2 \cap A_3 \neq \phi$.
$A_1 = \{1, 2\}$; $A_2 = \{3, 5\}$; $A_3 = \{4, 5, 6\}$.

The following result shows that if P is a partition of a set A, then P can be used to construct an equivalence relation on A.

Theorem: Let P be a partition of a set *A*. Define a relation *R* on *A* as *a R b* if and only if *a*, *b* belong to the same block of P then *R* is an equivalence relation on *A*.

Example 3.25: Consider the partition defined in Example 3.23. Then the equivalence relation as defined from the partition is:
R={(1, 1),(1, 2),(2, 1),(2, 2),(3, 3),(3, 5), (5, 3), (5, 5), (4, 4)}.

Now, we shall define equivalence classes of R on a set A.

Theorem: Let R be an equivalence relation on a set A and let $a, b \in A$, then $a \, R \, b$ if and only if $R(a) = R(b)$, where $R(a)$ is defined as: $R(a) = \{x \in A: a \, R \, x\}$. $R(a)$ is called as **relative set** of a.

Example 3.26: If we consider an example in 3.25, we observe that, $R(1) = R(2)$, $R(3) = R(5)$.

Because R (1) = {1,2}, R (2) = {1,2}, R (3) = {3,5}, R(5) = {3,5}.

Earlier, we have seen that, a partition defines an equivalence relation. Now, we shall see that, an equivalence relation defines a partition.

Theorem: Let R be an equivalence relation on A and let P be the collection of all distinct relative sets R(a) for a $\in$ A. Then P is a partition of A and R is equivalence relation of this partition.

Note: If R is an equivalence relation on A, then sets $R(a)$ are called as equivalence classes of R.

Example 3.27: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$. We observe that $R(1) = R(2)$ and $R(3) = R(4)$ and hence P = { {1, 2}, {3, 4} }.

Example 3.28: Let A = Z (set of integers) and define R as

$R = \{(a, b) \in A \times A: a \equiv b \pmod 5\}$. Then, we have,

$R(1) = \{......, -14, -9, -4, 1, 6, 11, ... \}$

$R(2) = \{......, -13, -8, -3, 2, 7, 12, ... \}$

$R(3) = \{......, -12, -7, -2, 3, 8, 13, ... \}$

$R(4) = \{......, -11, -6, -1, 4, 9, 14, ... \}$

$R(5) = \{......, -10, -5, 0, 5, 10, 15, ... \}$.

$R(1)$, $R(2)$, $R(3)$, $R(4)$ and $R(5)$ form partition on Z with respect to given equivalence relation.

3.10 UNIT END EXERCISE:

1. Show that we can have $A \cap B = A \cap C$, without $B = C$.

2. Prove that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.
(Note that, this can be used as a definition of $A \oplus B$)

3. Determine whether or not each of the following is a partition of the set N of natural numbers.
a.[$\{n : n > 5\}$, $\{n : n < 5\}$]
b.[$\{n : n > 5\}$, $\{0\}$, $\{1, 2, 3, 4, 5\}$]
c.[$\{n : n^2 > 11\}$, $\{n : n^2 < 11\}$]

4. Suppose $N = \{1, 2, 3, ..., \}$ is a universal set and
$A = \{x : x \le 6\}$, $B = \{x : 4 \le x \le 6\}$,
$C = \{1, 3, 5, 7, 9\}$, $D = \{2, 3, 5, 7, 8\}$
Find (i) $A \oplus B$ (ii) $B \oplus C$ (iii) $A \cap (B \oplus D)$
(iv) $(A \cap B) \oplus (A \cap D)$

5. Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by "x divides y", written an $x \mid y$.
   a. Write R as a set of ordered pairs.
   b. Draw a directed graph of R.
   c. Write down the matrix of relation R.
   d. Find the inverse relation R^{-1} of R and describe it in words.

6. Give an example of relations $A = \{1, 2, 3\}$ having the stated property.
   a. R is both symmetric and antisymmetric
   b. R is neither symmetric nor antisymmetric
   c. R is transitive but $R \cup R^{-1}$ is not transitive.

7. Let A be a set of non-zero integers and let $=$ be the relation on $A \times A$ defined by $(a, b) = (c, d)$, whenever $ad = bc$. Prove that $=$ is an equivalence relation.

8. Prove that if R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.